

CONFORMAL BOUNDS FOR THE FIRST EIGENVALUE OF THE p -LAPLACIAN

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ABSTRACT. : Let M be a compact, connected, m -dimensional manifold without boundary and $p > 1$. For $1 < p \leq m$, we prove that the first eigenvalue $\lambda_{1,p}$ of the p -Laplacian is bounded on each conformal class of Riemannian metrics of volume one on M . For $p > m$, we show that any conformal class of Riemannian metrics on M contains metrics of volume one with $\lambda_{1,p}$ arbitrarily large. As a consequence, we obtain that in two dimensions $\lambda_{1,p}$ is uniformly bounded on the space of Riemannian metrics of volume one if $1 < p \leq 2$, respectively unbounded if $p > 2$.

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1. INTRODUCTION

Let M be a compact m -dimensional manifold. All through this paper we will assume that M is connected and without boundary. The p -Laplacian ($p > 1$) associated to a Riemannian metric g on M is given by

$$\Delta_p u = \delta(|du|^{p-2} du),$$

where $\delta = -\operatorname{div}_g$ is the adjoint of d for the L^2 -norm induced by g on the space of differential forms. This operator can be viewed as an extension of the Laplace-Beltrami operator which corresponds to $p = 2$. The real numbers λ for which the nonlinear partial differential equation

$$\Delta_p u = \lambda |u|^{p-2} u$$

has nontrivial solutions are the *eigenvalues* of Δ_p , and the associated solutions are the *eigenfunctions* of Δ_p . Zero is an eigenvalue of Δ_p , the associated eigenfunctions being the constant functions. The set of the nonzero eigenvalues is a nonempty, unbounded subset of $(0, \infty)$ [6]. The infimum $\lambda_{1,p}$ of this set is itself a positive eigenvalue, the *first eigenvalue* of Δ_p , and has a Rayleigh type variational characterization [15]:

$$\lambda_{1,p}(M, g) = \inf \left\{ \frac{\int_M |du|^p \nu_g}{\int_M |u|^p \nu_g} \mid u \in W^{1,p}(M) \setminus \{0\}, \int_M |u|^{p-2} u \nu_g = 0 \right\},$$

where ν_g denotes the Riemannian volume element associated to g .

The first eigenvalue of Δ_p can be viewed as a functional on the space of Riemannian metrics on M :

$$g \mapsto \lambda_{1,p}(M, g).$$

Since $\lambda_{1,p}$ is not invariant under dilatations ($\lambda_{1,p}(M, cg) = c^{-\frac{p}{2}} \lambda_{1,p}(M, g)$), a normalization is needed when studying the uniform boundedness of this functional. It is common to restrict $\lambda_{1,p}$ to the set $\mathcal{M}(M)$ of Riemannian metrics of volume one on M . In the linear case $p = 2$ this problem has been extensively studied in various degrees of generality. The functional $\lambda_{1,2}$ was shown to be uniformly bounded on $\mathcal{M}(M)$ in two dimensions [7], [16], [8], and unbounded in three or more dimensions [13], [14], [12], [1], [2], [3]. However, $\lambda_{1,2}$ becomes uniformly bounded when restricted to any conformal class of Riemannian metrics in $\mathcal{M}(M)$ [4].

In the general case $p > 1$, the functional $\lambda_{1,p}$ is unbounded on $\mathcal{M}(M)$ in three or more dimensions [11]. In this paper we study the existence of uniform upper bounds for the restriction of $\lambda_{1,p}$ to conformal classes of Riemannian metrics in $\mathcal{M}(M)$:

- for $1 < p \leq m$ we extend the results from the linear case and obtain an explicit upper bound for $\lambda_{1,p}$ in terms of p , the dimension m and the Li-Yau n -conformal volume.
- for $p > m$, we consider first the case of the unit sphere S^m and we construct Riemannian metrics in $\mathcal{M}(S^m)$, conformal to the standard metric can and with $\lambda_{1,p}$ arbitrarily large. We use then the result on spheres to show that any conformal class of Riemannian metrics on M contains metrics of volume one with $\lambda_{1,p}$ arbitrarily large.

As a consequence, we obtain that in two dimensions, $\lambda_{1,p}$ is uniformly bounded on $\mathcal{M}(M)$ when $1 < p \leq 2$, and unbounded when $p > 2$.

2. THE CASE $1 < p \leq m$: LI-YAU TYPE UPPER BOUNDS

Let g be a Riemannian metric on M and denote by $[g] = \{fg \mid f \in C^\infty(M), f > 0\}$ the conformal class of g . Let $G(n) = \{\gamma \in \text{Diff}(S^n) \mid \gamma^*can \in [can]\}$ denote the group of conformal diffeomorphisms of (S^n, can) .

For n big enough, the Nash-Moser Theorem ensures (via the stereographic projection) that the set $I_n(M, [g]) = \{\phi : M \rightarrow S^n \mid \phi^*can \in [g]\}$ of conformal immersions from (M, g) to (S^n, can) is nonempty. The n -conformal volume of $[g]$ is defined by [8]:

$$V_n^c(M, [g]) = \inf_{\phi \in I_n(M, [g])} \sup_{\gamma \in G(n)} \text{Vol}(M, (\gamma \circ \phi)^*can),$$

where $\text{Vol}(M, (\gamma \circ \phi)^*can)$ denotes the volume of M with respect to the induced metric $(\gamma \circ \phi)^*can$. By convention, $V_n^c(M, [g]) = \infty$ if $I_n(M, [g]) = \emptyset$.

Theorem 2.1. *Let M be an m -dimensional compact manifold and $1 < p \leq m$. For any metric $g \in \mathcal{M}(M)$ and any $n \in \mathbb{N}$ we have*

$$\lambda_{1,p}(M, g) \leq m^{\frac{p}{2}}(n+1)^{|\frac{p}{2}-1|} V_n^c(M, [g])^{\frac{p}{m}}.$$

Remark 2.2. In the linear case $p = 2$, this result was proved by Li and Yau [8] for surfaces and by El Soufi and Ilias [4] for higher dimensional manifolds.

Remark 2.3. Theorem 2.1 gives an explicit upper bound for $\lambda_{1,p}$, $1 < p \leq m$, in the case of some particular manifolds: the sphere S^m , the real projective space $\mathbb{R}P^m$, the complex projective space $\mathbb{C}P^d$, the equilateral torus \mathbb{T}_{eq}^2 , the generalized Clifford torus $S^r \left(\sqrt{r/r+q} \right) \times S^q \left(\sqrt{q/r+q} \right)$, endowed with their canonical metrics.

For these manifolds we have [4]: $V_n^c(M, [can]) = Vol(M, \frac{\lambda_{1,2}}{m} can)$ for $n+1$ greater or equal to the multiplicity of $\lambda_{1,2}$.

Using the relationships between the conformal volume and the genus of a compact surface [5] we obtain:

Corollary 2.4. *Suppose $m = 2$ and $1 < p \leq 2$. Then for any metric $g \in \mathcal{M}(M)$*

$$\lambda_{1,p}(M, g) \leq k_p \left[\frac{\text{genus}(M) + 3}{2} \right]^{\frac{p}{2}},$$

where $[\cdot]$ denotes the integer part, $k_p = 3^{\lfloor \frac{p}{2} - 1 \rfloor} (8\pi)^{\frac{p}{2}}$ if M is orientable and $k_p = 5^{\lfloor \frac{p}{2} - 1 \rfloor} (24\pi)^{\frac{p}{2}}$ if not.

Remark 2.5. In the case $p = 2$ and $M = S^2$, this result is the well known Hersch inequality [7]. For higher genus surfaces, the upper bound of $\lambda_{1,2}$ in terms of the genus was obtained by El-Soufi and Ilias [5] by improving a previous result of Yang and Yau [16].

In order to prove Theorem 2.1 we need two Lemmas:

Lemma 2.6. *Let $\phi : (M, g) \rightarrow (S^n, can)$ be a smooth map whose level sets are of measure zero in (M, g) . Then for any $p > 1$ there exists $\gamma \in G(n)$ such that*

$$\int_M |(\gamma \circ \phi)_i|^{p-2} (\gamma \circ \phi)_i \nu_g = 0, \quad 1 \leq i \leq n+1.$$

Proof of Lemma 2.6. Let $a \in S^n$ and denote by π_a the stereographic projection of pole a . Let $t \in (0, 1]$ and $H_{\frac{1-t}{t}} = e^{\frac{1-t}{t}} \cdot Id_{\mathbb{R}^n}$ (i.e. $H_{\frac{1-t}{t}}$ is the linear dilatation of \mathbb{R}^n

of factor $e^{\frac{1-t}{t}}$). Let $\gamma_t^a \in G(n)$, $\gamma_t^a(x) = \begin{cases} \pi_a^{-1} \circ H_{\frac{1-t}{t}} \circ \pi_a(x) & \text{if } x \in S^n \setminus \{a\} \\ a & \text{if } x = a \end{cases}$

and consider the continuous map

$$F : (0, 1] \times S^n \rightarrow \mathbb{R}^{n+1}$$

$$F(t, a) = \frac{1}{Vol(M, g)} \left(\int_M |(\gamma_t^a \circ \phi)_1|^{p-2} (\gamma_t^a \circ \phi)_1 \nu_g, \dots, \int_M |(\gamma_t^a \circ \phi)_{n+1}|^{p-2} (\gamma_t^a \circ \phi)_{n+1} \nu_g \right).$$

For any $x \in M \setminus \{\phi^{-1}(-a)\}$ we have $\lim_{t \rightarrow 0+} \gamma_t^a \circ \phi(x) = a$. Since $\phi^{-1}(-a)$ is of measure zero in M , we can extend F into a continuous function on $[0, 1] \times S^n$ by setting

$$F(0, a) = (|a_1|^{p-2} a_1, \dots, |a_{n+1}|^{p-2} a_{n+1}).$$

The map $a \rightarrow F(0, a)$ is odd on S^n , and since $\gamma_1^a = Id_{S^n}$, the map $a \rightarrow F(1, a)$ is constant. Assume $\|F(t, a)\| \neq 0$ for any $(t, a) \in [0, 1] \times S^n$. Then the map

$$G : [0, 1] \times S^n \rightarrow S^n$$

$$G(t, a) = \frac{F(t, a)}{\|F(t, a)\|}$$

gives a homotopy between the odd map $a \rightarrow G(0, a)$ and the constant map $a \rightarrow G(1, a)$, and this is impossible. Hence there exists $(t, a) \in [0, 1] \times S^n$ such that $\|F(t, a)\| = 0$, i.e. $\int_M |(\gamma_t^a \circ \phi)_i|^{p-2} (\gamma_t^a \circ \phi)_i \nu_g = 0, \quad 1 \leq i \leq n+1.$ \square

Lemma 2.7. *Suppose $g \in \mathcal{M}(M)$ and let $\phi : (M, g) \rightarrow (S^n, \text{can})$ be a smooth map whose level sets are of measure zero in (M, g) . Then there exists $\gamma \in G(n)$ such that*

$$\lambda_{1,p}(M, g) \leq (n+1)^{|\frac{p}{2}-1|} \int_M |d(\gamma \circ \phi)|^p \nu_g,$$

where $|d(\gamma \circ \phi)|$ denotes the Hilbert-Schmidt norm of $d(\gamma \circ \phi)$.

Proof of Lemma 2.7. Lemma 2.6 implies there exists $\gamma \in G(n)$ such that $\psi = \gamma \circ \phi : M \rightarrow S^n$ verifies $\int_M |\psi_i|^{p-2} \psi_i \nu_g = 0$, $1 \leq i \leq n+1$. The variational characterization for $\lambda_{1,p}(M, g)$ implies that $\lambda_{1,p}(M, g) \leq \frac{\int_M |d\psi_i|^p \nu_g}{\int_M |\psi_i|^p \nu_g}$, $1 \leq i \leq n+1$.

Then

$$(2.1) \quad \lambda_{1,p}(M, g) \leq \frac{\int_M \sum_{i=1}^{n+1} |d\psi_i|^p \nu_g}{\int_M \sum_{i=1}^{n+1} |\psi_i|^p \nu_g}.$$

• *Case 1: $p \geq 2$.* It is straightforward that

$$(2.2) \quad \sum_{i=1}^{n+1} |d\psi_i|^p = \sum_{i=1}^{n+1} (|d\psi_i|^2)^{\frac{p}{2}} \leq \left(\sum_{i=1}^{n+1} |d\psi_i|^2 \right)^{\frac{p}{2}} = |d\psi|^p.$$

On the other hand

$$(2.3) \quad \sum_{i=1}^{n+1} |\psi_i|^p \geq (n+1)^{1-\frac{p}{2}} \left(\sum_{i=1}^{n+1} |\psi_i|^2 \right)^{\frac{p}{2}} = (n+1)^{1-\frac{p}{2}},$$

where we have used the fact that $x \rightarrow x^{\frac{p}{2}}$ is convex and that $\sum_{i=1}^{n+1} |\psi_i|^2 = 1$. Replacing (2.2) and (2.3) in (2.1) we obtain

$$\lambda_{1,p}(M, g) \leq (n+1)^{\frac{p}{2}-1} \int_M |d\psi|^p \nu_g.$$

• *Case 2: $1 < p < 2$.* Since $|\psi_i| \leq 1$ we have $|\psi_i|^2 \leq |\psi_i|^p$ and

$$(2.4) \quad 1 = \text{Vol}(M, g) = \int_M \sum_{i=1}^{n+1} |\psi_i|^2 \nu_g \leq \int_M \sum_{i=1}^{n+1} |\psi_i|^p \nu_g$$

On the other hand

$$(2.5) \quad \sum_{i=1}^{n+1} |d\psi_i|^p = \sum_{i=1}^{n+1} (|d\psi_i|^2)^{\frac{p}{2}} \leq (n+1)^{1-\frac{p}{2}} \left(\sum_{i=1}^{n+1} |d\psi_i|^2 \right)^{\frac{p}{2}} = (n+1)^{1-\frac{p}{2}} |d\psi|^p,$$

where the inequality follows from the concavity of $x \rightarrow x^{\frac{p}{2}}$. Replacing (2.4) and (2.5) in (2.1) we obtain

$$\lambda_{1,p}(M, g) \leq (n+1)^{1-\frac{p}{2}} \int_M |d\psi|^p \nu_g.$$

□

Proof of Theorem 2.1. Let $\phi : (M, g) \rightarrow (S^n, \text{can})$ be a conformal immersion. From Lemma 2.7 we have that there exists $\gamma \in G(n)$ such that

$$\lambda_{1,p}(M, g) \leq (n+1)^{|\frac{p}{2}-1|} \int_M |d(\gamma \circ \phi)|^p \nu_g.$$

Since $g \in \mathcal{M}(M)$, Hölder's inequality implies

$$\int_M |d(\gamma \circ \phi)|^p \nu_g \leq \left(\int_M |d(\gamma \circ \phi)|^m \nu_g \right)^{\frac{p}{m}}.$$

On the other hand since $\gamma \circ \phi : (M, g) \rightarrow (S^n, can)$ is a conformal immersion, $(\gamma \circ \phi)^* can = \frac{|d(\gamma \circ \phi)|^2}{m} g$ and we have

$$\int_M |d(\gamma \circ \phi)|^m \nu_g = m^{\frac{m}{2}} Vol(M, (\gamma \circ \phi)^* can) \leq m^{\frac{m}{2}} \sup_{\gamma \in G(n)} Vol(M, (\gamma \circ \phi)^* can).$$

Combining the inequalities above we obtain:

$$\lambda_{1,p}(M, g) \leq m^{\frac{p}{2}} (n+1)^{|\frac{p}{2}-1|} \left(\sup_{\gamma \in G(n)} Vol(M, (\gamma \circ \phi)^* can) \right)^{\frac{p}{m}}.$$

Taking the infimum over all $\phi \in I_n(M, [g])$ we obtain the desired inequality. \square

Proof of Corollary 2.4. In the case of surfaces, the n -conformal volume is bounded above by a constant depending only on the genus of the surface [5]. If M is orientable we have

$$V_n^c(M, [g]) \leq 4\pi \left\lceil \frac{genus(M) + 3}{2} \right\rceil \quad \text{for } n \geq 2.$$

If M is non orientable,

$$V_n^c(M, [g]) \leq 12\pi \left\lceil \frac{genus(M) + 3}{2} \right\rceil \quad \text{for } n \geq 4.$$

Theorem 2.1 implies now the desired result with $k_p = 3^{|\frac{p}{2}-1|} (8\pi)^{\frac{p}{2}}$ when M is orientable and $k_p = 5^{|\frac{p}{2}-1|} (24\pi)^{\frac{p}{2}}$ when M is non orientable. \square

3. THE CASE $p > m$

For the sake of self-containedness we include here the variational characterizations for the first eigenvalues for the Dirichlet and the Neumann problems for Δ_p . Let Ω be a domain in M and consider the Dirichlet problem:

$$\begin{cases} \Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

The infimum $\lambda_{1,p}^D(\Omega, g)$ of the set of eigenvalues for this problem is itself a positive eigenvalue with the variational characterization

$$\lambda_{1,p}^D(\Omega, g) = \inf \left\{ \frac{\int_{\Omega} |du|^p \nu_g}{\int_{\Omega} |u|^p \nu_g} \mid u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Consider now the Neumann problem on Ω :

$$\begin{cases} \Delta_p f &= |f|^{p-2} f \quad \text{in } \Omega \\ df(\eta) &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where η denotes the exterior unit normal vector field to $\partial\Omega$. Here too, the infimum $\lambda_{1,p}^N(\Omega, g)$ of the set of nonzero eigenvalues is a positive eigenvalue with the variational characterization

$$\lambda_{1,p}^N(\Omega, g) := \inf \left\{ \frac{\int_{\Omega} |df|^p \nu_g}{\int_{\Omega} |f|^p \nu_g} \mid f \in W^{1,p}(\Omega, g) \setminus \{0\}, \int_{\Omega} |f|^{p-2} f \nu_g = 0 \right\}.$$

We consider first the case of $(S^m, [can])$:

Theorem 3.1. *For any $p > m$, S^m carries Riemannian metrics of volume one, conformal to the standard metric can , with $\lambda_{1,p}$ arbitrarily large.*

Proof of Theorem 3.1. Let $r \in [0, \pi]$, denote the geodesic distance on (S^m, can) w.r.t. a point $x_0 \in S^m$. Let $\varepsilon > 0$ and define a radial function $f_\varepsilon : S^m \rightarrow \mathbb{R}$ by

$$(3.1) \quad f_\varepsilon(r) = \varepsilon^{\frac{4p}{m(p-m)}} \cdot \chi_{[0, \frac{\pi}{2}-\varepsilon] \cup [\frac{\pi}{2}+\varepsilon, \pi]}(r) + \chi_{(\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}+\varepsilon)}(r).$$

Let

$$\lambda_{1,p}(\varepsilon) = \inf \left\{ R_\varepsilon(u) := \frac{\int_{S^m} |du|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |u|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}} \mid u \in W^{1,p}(S^m) \setminus \{0\}, \right. \\ \left. \int_{S^m} |u|^{p-2} u |f_\varepsilon|^{\frac{m}{2}} \nu_{can} = 0 \right\}.$$

We will show first that

$$(3.2) \quad \limsup_{\varepsilon \rightarrow 0} \lambda_{1,p}(\varepsilon) \cdot \varepsilon^{\frac{p}{m}} = \infty.$$

Classical density arguments imply that there exists $u_\varepsilon \in W^{1,p}(S^m) \setminus \{0\}$ with $\int_{S^m} |u_\varepsilon|^{p-2} u_\varepsilon f_\varepsilon^{\frac{m}{2}} \nu_{can} = 0$ such that $\lambda_{1,p}(\varepsilon) = R_\varepsilon(u_\varepsilon)$. Let $\bar{u}_\varepsilon : S^m \rightarrow \mathbb{R}$ be a radial function defined by

$$(3.3) \quad \bar{u}_\varepsilon^p(r) = \frac{1}{V} \int_{S^{m-1}} |u_\varepsilon(r, \cdot)|^p \nu_{can}$$

where $V = Vol(S^{m-1}, can)$. Differentiating w.r.t. r we obtain

$$p \bar{u}_\varepsilon^{p-1} \bar{u}'_\varepsilon = \frac{p}{V} \int_{S^{m-1}} |u_\varepsilon|^{p-2} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \nu_{can}.$$

By Hölder's inequality we obtain

$$\bar{u}_\varepsilon^{p-1} |\bar{u}'_\varepsilon| \leq \frac{1}{V} \int_{S^{m-1}} |u_\varepsilon|^{p-1} \left| \frac{\partial u_\varepsilon}{\partial r} \right| \nu_{can} \leq \frac{1}{V} \left(\int_{S^{m-1}} |u_\varepsilon|^p \nu_{can} \right)^{\frac{p-1}{p}} \cdot \left(\int_{S^{m-1}} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^p \nu_{can} \right)^{\frac{1}{p}}.$$

It follows that

$$(3.4) \quad |\bar{u}'_\varepsilon|^p \leq \frac{1}{V} \int_{S^{m-1}} \left| \frac{\partial u_\varepsilon}{\partial r} \right|^p \nu_{can} \leq \frac{1}{V} \int_{S^{m-1}} |du_\varepsilon|^p \nu_{can}$$

On the other hand

$$\begin{aligned} \int_{S^m} |\bar{u}_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can} &= V \cdot \int_0^\pi |\bar{u}_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \sin r^{m-1} dr \\ &= \int_0^\pi \left[\int_{S^{m-1}} |u_\varepsilon|^p \nu_{can} \right] f_\varepsilon^{\frac{m}{2}} \sin r^{m-1} dr \\ &= \int_{S^m} |u_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}, \end{aligned}$$

where the second equality follows from (3.3). Similarly (3.4) implies

$$\int_{S^m} |\bar{u}'_\varepsilon|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can} \leq \int_{S^m} |du_\varepsilon|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can}.$$

In particular, we obtain that $\bar{u}_\varepsilon \in W^{1,p}(S^m)$ and

$$\begin{aligned} \lambda_{1,p}(\varepsilon) = R_\varepsilon(u_\varepsilon) &\geq \frac{\int_{S^m} |\bar{u}'_\varepsilon|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |\bar{u}_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}} \\ &\geq \min \left\{ \frac{\int_{S_+^m} |\bar{u}'_\varepsilon|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can}}{\int_{S_+^m} |\bar{u}_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}}, \frac{\int_{S_-^m} |\bar{u}'_\varepsilon|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can}}{\int_{S_-^m} |\bar{u}_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}} \right\}, \end{aligned}$$

where S_+^m, S_-^m denote the hemispheres centered at x_0 , respectively $-x_0$. Without loss of generality we may assume that

$$(3.5) \quad \lambda_{1,p}(\varepsilon) \geq \frac{\int_{S_+^m} |\bar{u}'_\varepsilon|^p f_\varepsilon^{\frac{m-p}{2}} \nu_{can}}{\int_{S_+^m} |\bar{u}_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}}.$$

Let $w_\varepsilon \in W^{1,p}(S_+^m)$, $w_\varepsilon = \begin{cases} \bar{u}_\varepsilon & \text{on } [0, \frac{\pi}{2} - \varepsilon] \\ \bar{u}_\varepsilon(\frac{\pi}{2} - \varepsilon) & \text{on } (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}] \end{cases}$ and $v_\varepsilon = \bar{u}_\varepsilon - w_\varepsilon$. Then $v_\varepsilon = 0$ on $[0, \frac{\pi}{2} - \varepsilon]$ and $w'_\varepsilon = 0$ on $(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$. Since v'_ε and w'_ε have disjoint supports, we have $|\bar{u}'_\varepsilon|^p = |v'_\varepsilon|^p + |w'_\varepsilon|^p$. On the other hand $|\bar{u}_\varepsilon|^p = |v_\varepsilon + w_\varepsilon|^p \leq 2^{p-1}(|v_\varepsilon|^p + |w_\varepsilon|^p)$. Then (3.5) and (3.1) imply

$$\begin{aligned} \lambda_{1,p}(\varepsilon) &\geq 2^{1-p} \frac{\int_{S_+^m} (|v'_\varepsilon|^p + |w'_\varepsilon|^p) f_\varepsilon^{\frac{m-p}{2}} \nu_{can}}{\int_{S_+^m} (|v_\varepsilon|^p + |w_\varepsilon|^p) f_\varepsilon^{\frac{m}{2}} \nu_{can}} \\ &= 2^{1-p} \frac{\int_{S_+^m} |v'_\varepsilon|^p \nu_{can} + \varepsilon^{-\frac{2p}{m}} \int_{S_+^m} |w'_\varepsilon|^p \nu_{can}}{\int_{S_+^m} |v_\varepsilon|^p \nu_{can} + \int_{S_+^m} |w_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can}} \end{aligned}$$

Quite to multiply \bar{u}_ε by a constant we may assume $\int_{S_+^m} |v_\varepsilon|^p \nu_{can} + \int_{S_+^m} |w_\varepsilon|^p f_\varepsilon^{\frac{m}{2}} \nu_{can} = 1$ and the inequality above becomes

$$(3.6) \quad \lambda_{1,p}(\varepsilon) \geq 2^{1-p} \int_{S_+^m} |v'_\varepsilon|^p \nu_{can} + \varepsilon^{-\frac{2p}{m}} \int_{S_+^m} |w'_\varepsilon|^p \nu_{can}$$

• *Case 1:* $\limsup_{\varepsilon \rightarrow 0} \int_{S_+^m} |w'_\varepsilon|^p \nu_{can} > 0$.

Inequality (3.6) implies that $\lambda_{1,p}(\varepsilon) \varepsilon^{\frac{p}{m}} \geq 2^{1-p} \varepsilon^{-\frac{p}{m}} \int_{S_+^m} |w'_\varepsilon|^p \nu_{can}$, and therefore (3.2) is verified.

• *Case 2:* $\lim_{\varepsilon \rightarrow 0} \int_{S_+^m} |w'_\varepsilon|^p \nu_{can} = 0$.

Then we may find a sequence $\varepsilon_n \rightarrow 0$ such that $w_{\varepsilon_n} \rightarrow c$ strongly in $L^p(M)$, where c is a constant. In particular since $p > m$, $\{f_{\varepsilon_n}\}$ is uniformly bounded and we have $\lim_{n \rightarrow \infty} \int_{S_+^m} f_{\varepsilon_n}^{\frac{m}{2}} \nu_{can} = 0$. It follows that $\lim_{n \rightarrow \infty} \int_{S_+^m} |w_{\varepsilon_n}|^p f_{\varepsilon_n}^{\frac{m}{2}} \nu_{can} = \lim_{n \rightarrow \infty} \int_{S_+^m} (|w_{\varepsilon_n}|^p - |c|^p) f_{\varepsilon_n}^{\frac{m}{2}} \nu_{can} + |c|^p \lim_{n \rightarrow \infty} \int_{S_+^m} f_{\varepsilon_n}^{\frac{m}{2}} \nu_{can} = 0$. Hence for ε_n

small enough, $\int_{S_+^m} |v_{\varepsilon_n}|^p \nu_{can} = 1 - \int_{S_+^m} |w_{\varepsilon_n}|^p f_{\varepsilon_n}^{\frac{m}{2}} \nu_{can} \geq \frac{1}{2}$ and (3.6) implies

$$(3.7) \quad \begin{aligned} \lambda_{1,p}(\varepsilon_n) &\geq 2^{1-\frac{p}{2}} \int_{S_+^m} |v'_{\varepsilon_n}|^p \nu_{can} \geq 2^{-\frac{p}{2}} \frac{\int_{S_+^m} |v'_{\varepsilon_n}|^p \nu_{can}}{\int_{S_+^m} |v_{\varepsilon_n}|^p \nu_{can}} = 2^{-\frac{p}{2}} \frac{\int_{\frac{\pi}{2}-\varepsilon_n}^{\frac{\pi}{2}} |v'_{\varepsilon_n}|^p \sin r^{m-1} dr}{\int_{\frac{\pi}{2}-\varepsilon_n}^{\frac{\pi}{2}} |v_{\varepsilon_n}|^p \sin r^{m-1} dr} \\ &\geq 2^{-\frac{p}{2}} [\sin(\frac{\pi}{2} - \varepsilon_n)]^{m-1} \frac{\int_{\frac{\pi}{2}-\varepsilon_n}^{\frac{\pi}{2}} |v'_{\varepsilon_n}|^p dr}{\int_{\frac{\pi}{2}-\varepsilon_n}^{\frac{\pi}{2}} |v_{\varepsilon_n}|^p dr}. \end{aligned}$$

Let $\bar{v}_{\varepsilon_n} \in W_0^{1,p}(-\varepsilon_n, \varepsilon_n)$ be an even function such that $\bar{v}_{\varepsilon_n}(s) = v_{\varepsilon_n}(s + \frac{\pi}{2} - \varepsilon_n)$ for $0 \leq s \leq \varepsilon_n$. We have then

$$(3.8) \quad \frac{\int_{\frac{\pi}{2}-\varepsilon_n}^{\frac{\pi}{2}} |v'_{\varepsilon_n}|^p dr}{\int_{\frac{\pi}{2}-\varepsilon_n}^{\frac{\pi}{2}} |v_{\varepsilon_n}|^p dr} = \frac{\int_0^{\varepsilon_n} |\bar{v}'_{\varepsilon_n}|^p dr}{\int_0^{\varepsilon_n} |\bar{v}_{\varepsilon_n}|^p dr} = \frac{\int_{-\varepsilon_n}^{\varepsilon_n} |\bar{v}'_{\varepsilon_n}|^p dr}{\int_{-\varepsilon_n}^{\varepsilon_n} |\bar{v}_{\varepsilon_n}|^p dr} \geq \lambda_{1,p}^D(-\varepsilon_n, \varepsilon_n) = \varepsilon_n^{-p} \lambda_{1,p}^D(-1, 1).$$

Inequalities (3.7), (3.8) imply $\lambda_{1,p}(\varepsilon_n) \geq \varepsilon_n^{-p} \lambda_{1,p}^D(-1, 1)$ and (3.2) is verified again.

Fix now $\varepsilon > 0$ and let $\tilde{f}_{\varepsilon} \in C^\infty(S^m)$, radial with respect to x_0 and such that $\tilde{f}_{\varepsilon} \leq f_{\varepsilon}$, $\tilde{f}_{\varepsilon}(r) = f_{\varepsilon}(r) = 1$ on $[\frac{\pi}{2} - \frac{\varepsilon}{2}, \frac{\pi}{2} + \frac{\varepsilon}{2}]$ and $\tilde{f}_{\varepsilon}(\pi - r) = \tilde{f}_{\varepsilon}(r)$. Then

$$(3.9) \quad \begin{aligned} Vol(S^m, \tilde{f}_{\varepsilon} can) &= \int_{S^m} \tilde{f}_{\varepsilon}^{\frac{m}{2}} \nu_{can} = \int_{S^{m-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{f}_{\varepsilon}^{\frac{m}{2}} \sin r^{m-1} dr \nu_{can} \\ &> V \int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}+\frac{\varepsilon}{2}} \sin r^{m-1} dr \\ &> \varepsilon V [\sin(\frac{\pi}{2} - \varepsilon)]^{m-1}, \quad \text{where } V = Vol(S^{m-1}, can). \end{aligned}$$

We will compare now $\lambda_{1,p}(S^m, \tilde{f}_{\varepsilon} can)$ and $\lambda_{1,p}(\varepsilon)$. Let \tilde{u}_{ε} be an eigenfunction for $\lambda_{1,p}(S^m, \tilde{f}_{\varepsilon} can)$ and denote by $\tilde{u}_{\varepsilon}^+, \tilde{u}_{\varepsilon}^-$ the positive, respectively, the negative part of \tilde{u}_{ε} . Then [9]

$$\lambda_{1,p}(S^m, \tilde{f}_{\varepsilon} can) = \frac{\int_{S^m} |d\tilde{u}_{\varepsilon}^+|^p \tilde{f}_{\varepsilon}^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |\tilde{u}_{\varepsilon}^+|^p \tilde{f}_{\varepsilon}^{\frac{m}{2}} \nu_{can}} = \frac{\int_{S^m} |d\tilde{u}_{\varepsilon}^-|^p \tilde{f}_{\varepsilon}^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |\tilde{u}_{\varepsilon}^-|^p \tilde{f}_{\varepsilon}^{\frac{m}{2}} \nu_{can}}$$

Let $t \in \mathbb{R}$ and $\tilde{u}_{\varepsilon,t} = t\tilde{u}_{\varepsilon}^+ + \tilde{u}_{\varepsilon}^-$. Then there is t_0 such that $\int_{S^m} |\tilde{u}_{\varepsilon,t_0}|^{p-2} \tilde{u}_{\varepsilon,t_0} \tilde{f}_{\varepsilon}^{\frac{m}{2}} \nu_{can} = 0$ and the equation above implies

$$(3.10) \quad \lambda_{1,p}(S^m, \tilde{f}_{\varepsilon} can) = \frac{\int_{S^m} |d\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_{\varepsilon}^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_{\varepsilon}^{\frac{m}{2}} \nu_{can}} \geq \frac{\int_{S^m} |d\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_{\varepsilon}^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |\tilde{u}_{\varepsilon,t_0}|^p \tilde{f}_{\varepsilon}^{\frac{m}{2}} \nu_{can}} \geq \lambda_{1,p}(\varepsilon),$$

where the first inequality follows from the fact that $\tilde{f}_{\varepsilon} \leq f_{\varepsilon}$ and the second from the variational characterization for $\lambda_{1,p}(\varepsilon)$. Inequalities (3.9), (3.10) and (3.2) yield

$$\limsup_{\varepsilon \rightarrow 0} \lambda_{1,p}(S^m, \tilde{f}_{\varepsilon} can) Vol(S^m, \tilde{f}_{\varepsilon} can)^{\frac{p}{m}} \geq V^{\frac{p}{m}} \cdot \limsup_{\varepsilon \rightarrow 0} \lambda_{1,p}(\varepsilon) \cdot \varepsilon^{\frac{p}{m}} = \infty.$$

Finally, let $h_{\varepsilon} = Vol(S^m, \tilde{f}_{\varepsilon} can)^{-\frac{2}{m}} \tilde{f}_{\varepsilon}$. We have then

$$Vol(S^m, h_{\varepsilon} can) = 1 \quad \text{and} \quad \limsup_{\varepsilon \rightarrow 0} \lambda_{1,p}(S^m, h_{\varepsilon} can) = \infty. \quad \square$$

We will extend the construction from $(S^m, [can])$ to $(M, [g])$ by means of the first eigenvalue for the Neumann problem for Δ_p on a domain Ω in M .

Theorem 3.2. *Let (M, g) be a compact Riemannian manifold of dimension m . Then for any $p > m$, $[g]$ contains Riemannian metrics of volume one with $\lambda_{1,p}$ arbitrarily large.*

Proof of Theorem 3.2. Let r denote the geodesic distance on (S^m, can) w.r.t. a point x_0 . Let $f \in C^\infty(S^m)$ be a function radial w.r.t. x_0 , such that $f(r) = f(\pi - r)$ and $Vol(S^m, fcan) = 1$. As before, let S_+^m denote the hemisphere centered at x_0 . Let v be an eigenfunction for $\lambda_{1,p}^N(S_+^m, fcan)$ and let $w \in W^{1,p}(S^m)$, $w(r) = \begin{cases} v(r) & \text{if } 0 \leq r \leq \frac{\pi}{2} \\ v(\pi - r) & \text{if } \frac{\pi}{2} < r \leq \pi \end{cases}$. Then $\int_{S^m} |w|^{p-2} w f^{\frac{m-p}{2}} \nu_{can} = 2 \int_{S_+^m} |v|^{p-2} v f^{\frac{m}{2}} \nu_{can} = 0$ and the variational characterization for $\lambda_{1,p}(S^m, fcan)$ implies

$$(3.11) \quad \lambda_{1,p}(S^m, fcan) \leq \frac{\int_{S^m} |dw|^p f^{\frac{m-p}{2}} \nu_{can}}{\int_{S^m} |w|^p f^{\frac{m}{2}} \nu_{can}} = \frac{\int_{S_+^m} |dv|^p f^{\frac{m-p}{2}} \nu_{can}}{\int_{S_+^m} |v|^p f^{\frac{m}{2}} \nu_{can}} = \lambda_{1,p}^N(S_+^m, fcan)$$

Let Ω be a domain in M such that there exists a diffeomorphism $\Phi : \Omega \rightarrow S_+^m$. We may assume Ω is included in the open region of a local chart of M . In this chart we have $\nu_g = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$ and $\nu_{\Phi^*can} = \sqrt{\det((\Phi^*can)_{ij})} dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$. There exist positive constants c_1, c_2 such that

$$(3.12) \quad c_1 \sqrt{\det(g_{ij})} \leq \sqrt{\det((\Phi^*can)_{ij})} \leq c_2 \sqrt{\det(g_{ij})} \quad \text{on } \Omega.$$

We will compare now $\lambda_{1,p}^N(S_+^m, fcan)$ and $\lambda_{1,p}^N(\Omega, (f \circ \Phi)g)$. Note first that since Φ is an isometry between $(\Omega, (f \circ \Phi)\Phi^*can)$ and $(S_+^m, fcan)$ we have

$$(3.13) \quad \lambda_{1,p}^N(S_+^m, fcan) = \lambda_{1,p}^N(\Omega, (f \circ \Phi)\Phi^*can)$$

Let u be an eigenfunction for $\lambda_{1,p}^N(\Omega, (f \circ \Phi)g)$ and denote by u^+, u^- the positive, respectively, the negative part of u . Then there is $s \in \mathbb{R}$ such that the function $u_s = su^+ + u^-$ verifies $\int_{\Omega} |u_s|^{p-2} u_s (f \circ \Phi)^{\frac{m-p}{2}} \nu_{\Phi^*can} = 0$. Furthermore

$$(3.14) \quad \begin{aligned} \lambda_{1,p}^N(\Omega, (f \circ \Phi)g) &= \frac{\int_{\Omega} |du_s|^p (f \circ \Phi)^{\frac{m-p}{2}} \nu_g}{\int_{\Omega} |u_s|^p (f \circ \Phi)^{\frac{m}{2}} \nu_g} \geq \frac{c_1 \int_{\Omega} |du_s|^p (f \circ \Phi)^{\frac{m-p}{2}} \nu_{\Phi^*can}}{c_2 \int_{\Omega} |u_s|^p (f \circ \Phi)^{\frac{m}{2}} \nu_{\Phi^*can}} \\ &\geq \frac{c_1}{c_2} \lambda_{1,p}^N(\Omega, (f \circ \Phi)\Phi^*can), \end{aligned}$$

where the first inequality follows from (3.12) and the second from the variational characterization of $\lambda_{1,p}^N(\Omega, (f \circ \Phi)\Phi^*can)$. From (3.11), (3.13) and (3.14) we obtain

$$(3.15) \quad \lambda_{1,p}^N(\Omega, (f \circ \Phi)g) \geq \frac{c_1}{c_2} \lambda_{1,p}(S^m, fcan).$$

Let now $\delta > 0$; there is an extension $\widetilde{f \circ \Phi}$ of $f \circ \Phi$ on the entire manifold M such that the metric $\tilde{g} = \widetilde{f \circ \Phi} g$ verifies [10]: $\lambda_{1,p}(M, \tilde{g}) > \lambda_{1,p}^N(\Omega, (f \circ \Phi)g) - \delta$. Inequality (3.15) implies

$$(3.16) \quad \lambda_{1,p}(M, \tilde{g}) > \frac{c_1}{c_2} \lambda_{1,p}(S^m, fcan) - \delta$$

On the other hand

$$(3.17) \quad \begin{aligned} Vol(M, \tilde{g}) &> Vol(\Omega, (f \circ \Phi)g) \geq \frac{1}{c_2} Vol(\Omega, (f \circ \Phi)\Phi^* can) \\ &= \frac{1}{c_2} Vol(S_+^m, f can) = \frac{1}{2c_2} Vol(S^m, f can) = \frac{1}{2c_2}. \end{aligned}$$

Let $K > 0$; from the proof of Theorem 3.1 we may assume that f is chosen such that $\lambda_{1,p}(S^m, f can) > 2^{\frac{p}{m}+1} c_1^{-1} c_2^{\frac{p}{m}+1} K$. For δ small enough such that $(2c_2)^{-\frac{p}{m}} \delta < K$, inequalities (3.16) and (3.17) imply

$$\lambda_{1,p}(M, \tilde{g}) Vol(M, \tilde{g})^{\frac{p}{m}} \geq [\frac{c_1}{c_2} \lambda_{1,p}(S^m, f can) - \delta] (2c_2)^{-\frac{p}{m}} > K.$$

Finally, let $h = Vol(M, \tilde{g})^{-\frac{2}{m}} \tilde{g}$. Then $h \in [g]$, $Vol(M, h) = 1$ and $\lambda_{1,p}(M, h) > K$. \square

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